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Extension of q-deformed analysis and q-deformed models of classical mechanics

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Abstract. In this paper we present the generalization of the q-derivative and investigate its properties. The formula of conjugation for scalar products with respect to the Lebesgue and discrete measures is obtained. This formalism is applied to classical mechanical systems depending on functional derivatives. We derive the deformed Euler-Lagrange equations and deformed Poisson bracket by an assumption that it determines the functional evolution of the systems. It appears that in such models a Hamiltonian analogue is no longer conserved in time. For a few examples the explicit forms of constants of evolution are given.

1. Introduction

Recently various aspects of q-deformed algebras [1, 2] were considered. The annihilation and creation operators of independent q-oscillators, first defined in papers by Biedenharn [3] and MacFarlane [4] became a useful tool in the construction of deformed Lie algebras [5-7]. These new mathematical objects were applied to physical models, e.g. in the vertex and spin models [8, 9], and quantum optics [10]. On the other hand, certain equations including deformed operators were investigated and their exact solutions and spectra were obtained [11, 12].

The natural question arises; how to deal with classical systems depending on the q-derivative of variables and how to describe the time evolution of such a system. Here we should mention the paper by Caldi [13] in which the evolution of observables determined by the quantum deformed bracket, as well as q-evolution with respect to the non-deformed quantum bracket, were described.

In our work we shall deal with classical models and assume that the action includes q-derivative of variables. We derive the necessary condition for the stationary point of such functional, e.g. the analogue of Euler-Lagrange equations. Assuming that the deformed Poisson bracket should determine q-evolution of functions on phase-space, we obtain the explicit formula for the deformed Poisson bracket and discuss its properties. It appears that, similarly to the commutation properties of deformed $SU_q(n)$ algebras, it is neither symmetric nor anti-symmetric.

The paper is organized as follows. In section 2 we give the extension of q-derivative to the new operator which we call the ϕ-derivative. In the following we calculate the inverse and conjugate operator and investigate their properties. In section 3.1 we consider classical models with discrete time, deriving for them Euler-Lagrange equations and deformed Poisson bracket. In section 3.2 analogous calculations have been performed for classical deformed models with continuous time.

2. q-analysis and its extension

2.1. Definitions

In this section we shall generalize the q-derivatives that are known and used in the realization of quantum algebras, given by the formulae

$$\partial_q f(\tau) := \frac{f(q^2\tau) - f(\tau)}{q^2\tau - \tau} = (q^2\tau - \tau)^{-1}[\zeta_{q^2} - 1]f(\tau) \tag{1}$$

$$\partial_q := (q^2\tau - \tau)^{-1}[\zeta_{q^2} - 1] \qquad \bar{\partial}_q := (q\tau - q^{-1}\tau)^{-1}[\zeta_q - \zeta_{q^{-1}}] \tag{2}$$

where we have introduced a dilation operator acting on the function as follows

$$\zeta_a f(\tau) := f(a\tau).$$

Formula (1) suggests the following expression for the general functional ϕ-derivative of the real function

$$\partial_\phi f(\tau) := \frac{f(\phi(\tau)) - f(\tau)}{\phi(\tau) - \tau} \tag{3}$$

with ζ_ϕ operator of the form

$$\zeta_\phi f(\tau) := f(\phi(\tau)).$$

Taking the function ϕ as

$$\phi(\tau) = q^2\tau$$

one gets the q-derivative, while translation given by the formula

$$\phi(\tau) = \tau + h$$

produces the difference derivative also used in functional evolution describing muon mechanics [14].

2.2. Properties of ϕ-differentiation: its inverse and conjugated operator

Let us now consider the properties of the defined operator. First we check how it acts on simple functions of real variables.

On monomials we get

$$\partial_\phi \tau^n = \sum_{k=0}^{n-1} \tau^k [\phi(\tau)]^{n-1-k} \tag{4}$$

with the commutation formula similar to that of the q-derivative

$$\partial_\phi \tau = 1 + \phi(\tau)\partial_\phi. \tag{5}$$

The formula for exponential function can be easily obtained from the above expression for monomials

$$\partial_\phi \exp(\tau) = \exp_\phi(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-1} \tau^k [\phi(\tau)]^{n-1-k}. \tag{6}$$

Negative powers give, under ϕ-differentiation, the following function:

$$\partial_\phi [1/\tau^n] = -\sum_{k=1}^{n} \frac{1}{\tau^k [\phi(\tau)]^{n-k+1}}. \tag{7}$$

One can check that ϕ-derivative has properties similar to that of differentiation procedure, namely:

(i) The ϕ-derivative is a linear operator

$$\partial_\phi[\alpha f+\beta g](\tau)=\alpha\partial_\phi f(\tau)+\beta\partial_\phi g(\tau).$$

(ii) The ϕ-derivative gives, for products of functions, the slightly modified formula

$$\partial_\phi f\cdot g[\tau]=\partial_\phi f[\tau]g[\tau]+f[\phi(\tau)]\partial_\phi g[\tau]. \tag{8}$$

(iii) For the ϕ-derivative of composed function we get

$$\partial_\phi f\circ g[\tau]=\partial_{g\circ\phi\circ g^{-1}}f[g(\tau)]\partial_\phi g[\tau]. \tag{9}$$

(iv) Formula (9) implies the following expression for the ϕ-derivative of the inverse function

$$\partial_{g\circ\phi\circ g^{-1}}q^{-1}[g(\tau)]=[\partial_\phi g[\tau]]^{-1}.$$

The inverse operator acting analogously to the definite integrals for q-derivatives is known:

$$\text{if } q<1 \qquad \int_0^t u(\tau)\,\mathrm{d}\mu_q(\tau)=-\sum_{n=0}^{\infty}(q^{2n+2}t-q^{2n}t)u(q^{2n}t) \tag{10}$$

$$\text{if } q>1 \qquad \int_\infty^t u(\tau)\,\mathrm{d}\mu_q(\tau)=-\sum_{n=0}^{\infty}(q^{2n+2}t-q^{2n}t)u(q^{2n}t) \tag{11}$$

and for symmetric derivatives

$$\text{if } q<1 \qquad \int_0^t u(\tau)\,\mathrm{d}\mu_q(\tau)=-\sum_{n=0}^{\infty}(q^{2n+2}t-q^{2n}t)u(q^{2n+1}t)$$

$$\text{if } q>1 \qquad \int_\infty^t u(\tau)\,\mathrm{d}\mu_q(\tau)=-\sum_{n=0}^{\infty}(q^{2n+2}t-q^{2n}t)u(q^{2n+1}t).$$

It is possible to construct expressions like this for ϕ-derivatives provided that the function ϕ satisfies the conditions—the following limits (finite or not) should exist

$$a=\lim_{n\to\infty}\phi^n(\tau) \qquad b=\lim_{n\to-\infty}\phi^n(\tau) \qquad \forall n\in N \qquad \phi^n\neq 1.$$

Let us note that neither of the iterations of function ϕ, the condition that coincide with the unity operator, plays a crucial role in the construction of the ϕ integral. As examples of such functions we can take $\phi(\tau)=1/\tau$ or $\phi(\tau)=-\tau$. Considering extension of our construction to complex variables we see that for a q-derivative, with q equal to one of roots of unity $q^n=1$, we would not have been able to build the inverse operator.

Taking these restrictions into account we can obtain the formula for the ϕ-integral after iterating the formula (3) for ϕ-derivative and summing up:

$$\int_a^t u(\tau)\,\mathrm{d}\mu_\phi(\tau)=-\sum_{n=0}^{\infty}(\phi^{n+1}(t)-\phi^n(t))u(\phi^n(t)) \tag{12}$$

$$\int_t^b u(\tau)\,\mathrm{d}\mu_\phi(\tau)=-\sum_{n=-1}^{-\infty}(\phi^{n+1}(t)-\phi^n(t))u(\phi^n(t)). \tag{13}$$

The constructed operator has the following properties:

(i) It is linear.

(ii) The integration can be done by parts according to the formula (8) for the ϕ-derivative

$$\int_{t_0}^{t_1} \partial_\phi f[s]g[s]\, \mathrm{d}\mu_\phi(s) = f\cdot g[s]|_{t_0}^{t_1} - \int_{t_0}^{t_1} f[\phi(s)]\partial_\phi g[s]\, \mathrm{d}\mu_\phi(s). \tag{14}$$

(iii) After change of variables we get

$$\int_{t_0}^{t_1} f(g(s))\partial_\phi g[s]\, \mathrm{d}\mu_\phi(s) = \int_{g(t_0)}^{g(t_1)} f(g(s))\, \mathrm{d}\mu_{g\circ\phi\circ g^{-1}}(g(s)). \tag{15}$$

The important problem in applications of the proposed formalism is the conjugated operator of the ϕ-derivative. We have two possibilities. One is to find out whether it could be conjugated for scalar products given by the Lebesgue measure on real, positive numbers. The other way is construction of a scalar product from an inverse operator of the ϕ-derivative. Such an operator applied to the formula of ϕ-differentiation of product of functions, gives the formula of ϕ-integration by parts. Thus, setting functions describing physical states equal to zero at the boundaries, we shall construct the conjugated ϕ-derivative as in the real non-deformed analysis. The further application to mechanics produces two sets of models with continuous and discrete time, which will be the subject of the next section.

First we consider the scalar product given by the integral connected with ϕ-derivative

$$\begin{aligned}\langle f|g\rangle_\phi &= \int_{t_0}^{t_1} f(s)g(s)\, \mathrm{d}\mu_\phi(s)\\ &= -\sum_{n=0}^{\infty} (\phi^{n+1}(t_1)-\phi^n(t_1))f(\phi^n(t_1))g(\phi^n(t_1))\\ &\quad + \sum_{n=0}^{\infty} (\phi^{n+1}(t_0)-\phi^n(t_0))f(\phi^n(t_0))g(\phi^n(t_0))\end{aligned} \tag{16}$$

where $a \leqslant t_0 < t_1 \leqslant b$.

The conjugated q-derivative was used by us in the realization of harmonic oscillators as operators acting on space of functions of real variables [15].

Analysing the general definition (16) we note that the product is degenerate—giving zero norm for non-vanishing functions. To obtain the non-degenerate form of the scalar product we must pass from $\mathfrak{R}[t_0, t_1]$, the space of real functions on $[t_0, t_1]$, to a quotient space $\mathfrak{R}[t_0, t_1]/\approx$ where the equivalence relation is defined by the formula

$$f \approx g \Leftrightarrow \langle f-g|f-g\rangle_\phi = 0.$$

Using the formula (14) for integration by parts and restricting ourselves to the functions from $\mathfrak{R}[t_0, t_1]/\approx$ which vanish for t_0 and t_1 we get the conjugated operators

$$\partial_\phi^+ = -\partial_\phi \circ \zeta_{\phi^{-1}}.$$

The models with continuous time require another definition of scalar product. However, for integrals with Lebesgue measure the q- and ϕ-derivatives do not fulfil the integration by parts formula. Therefore we must restrict ourselves to integrals over infinite intervals and apply a change of variables. Still we were only able to build a conjugated operator for the ϕ function taken as

$$\phi(\tau) = a\tau + b.$$

We should point out that the formulae for operators conjugated with respect to ϕ-integral and the Lebesgue measure coincide with these ϕ-functions for which both can be constructed. Therefore, as we will show, for the models with functional evolution the deformed Poisson brackets are the same for models with discrete and continuous time.

Let us note that the symmetric q-derivative $\bar{\partial}_q$ multiplied by an irrational number i is self-conjugated with respect to Lebesgue measure. This property, similar to self-conjugated ordinary derivative, differs the symmetric q-derivative from q-derivative (1) and its generalization. Since such operators give anti-symmetric Poisson brackets we consider, in the following, non-symmetric derivatives.

The conjugated non-symmetric ϕ-derivative can be expressed as a nonlinear function of function ϕ and the ϕ-derivative

$$\partial_\phi^+ = -\partial_\phi \sum_{n=0}^{\infty} (-1)^n [(\phi-1)\partial_\phi]^n. \tag{17}$$

It is also important to check the commutation relations of the ϕ-derivative and its conjugation

$$\partial_\phi^+ \partial_\phi = \frac{\phi^2-\phi}{\phi-1} \partial_\phi \partial_\phi^+ + \frac{\phi^2-\phi}{(\phi-1)^2} \partial_\phi^+. \tag{18}$$

One can see that the last term in the above formula disappears when the function ϕ fulfils the functional equation

$$\frac{\tau-\phi^{-1}(\tau)}{\phi(\tau)-\tau} = \text{const} = a^{-1} \qquad \phi^2(\tau)-\phi(\tau) = a\phi(\tau)-a\tau. \tag{19}$$

This is the equation of linear iteration of order 2 [16] and one can check that linear function

$$\phi(\tau) = a\tau + b \tag{20}$$

is a solution for arbitrary values of parameter b.

For such functions we have also the following commutation formula for the ϕ-derivative and ζ_ϕ operator

$$\partial_\phi \zeta_{\phi^{-1}} f(\tau) = a^{-1} \zeta_{\phi^{-1}} \partial_\phi f(\tau) \tag{21}$$

$$\partial_\phi \zeta_{\phi^n} f(\tau) = a^n \zeta_{\phi^n} \partial_\phi f(\tau). \tag{22}$$

2.3. Application of ϕ-derivative formalism to certain functional equations

To close the first part of our paper we present an application of analysis based on the ϕ-derivative idea applied to a functional equation with separate variables

$$\partial_q f(\tau) = u[f(\tau)]\psi(\tau) \qquad q<1. \tag{23}$$

It can be written down as the following integral equation

$$\begin{aligned} &\int_a^{f(\tau)} [u(s)]^{-1}\, \mathrm{d}\mu_{f\circ\kappa_{q^2}\circ f^{-1}}(s) = \int_0^\tau \psi(s)\, \mathrm{d}\mu_q(s) \\ &\kappa_{q^2} f(\tau) = q^2 f(\tau). \end{aligned} \tag{24}$$

When the integral with respect to the $f\circ\kappa_{q^2}\circ f^{-1}$ functional measure can be performed we get the solution of (23) in an explicit form. We list below a few nonlinear iterative functional equations and their solutions satisfying the initial condition $f(0)=\alpha$ obtained via the $f\circ\kappa_{q^2}\circ f^{-1}$ integral

$$\partial_q f(\tau)=f(\tau)f(q^2\tau)\psi(\tau)$$
$$f(\tau)=\left[\frac{1}{\alpha}-\int_0^{\tau}\psi(s)\,\mathrm{d}\mu_q(s)\right]^{-1} \tag{25}$$

$$\partial_q f(\tau)=\left[\sum_{k=0}^{n}[f(\tau)]^k[f(q^2\tau)]^{n-k}\right]^{-1}\psi(\tau)$$
$$f(\tau)=\left[\alpha^{n+1}+\int_0^{\tau}\psi(s)\,\mathrm{d}\mu_q(s)\right]^{1/(n+1)} \tag{26}$$

$$\partial_q f(\tau)=\left[\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}[f(\tau)]^k[f(q^2\tau)]^{n-k-1}\right]^{-1}\psi(\tau)$$
$$f(\tau)=\ln\left[\exp(\alpha)+\int_0^{\tau}\psi(s)\,\mathrm{d}\mu_q(s)\right]=\alpha\ln\left[\int_0^{\tau}\psi(s)\,\mathrm{d}\mu_q(s)\right] \tag{27}$$

$$\partial_q f(\tau)=\left[-\sum_{k=1}^{n}\frac{1}{[f(\tau)]^k[f(q^2\tau)]^{n-k+1}}\right]^{-1}\psi(\tau)$$
$$f(\tau)=\left[\frac{1}{\alpha^n}+\int_0^{\tau}\psi(s)\,\mathrm{d}\mu_q(s)\right]^{-1/n}. \tag{28}$$

As we have shown the general ϕ-derivative and integral calculus allows us to solve some nonlinear functional equations and probably could also be used for a wider class of equations.

In this section we have given a brief review of formulae for analysis based on the q- and ϕ-derivative concept. Our further aim is to apply this formalism to mechanics where functional evolution would be considered.

3. q-classical mechanics based on functional evolution

In the preceding section we investigated properties of general functional derivative. Here we shall use this formalism in mechanical systems which depend on functional derivatives of variables. One should mention that the functional derivatives, namely the difference- and q-derivatives, were applied before, both in context of theory of functional equations as well as in analysis of various physical systems.

However, functional derivatives were then used on the level of quantum mechanics—in the Schrödinger equation or in functional evolution. The last problem was studied in a paper by Caldi [13] who considered the following ways of time-evolution:

(1) $\mathrm{d}/(\mathrm{d}t)O=[O,H]_q$, where evolution is determined by a usual derivative and deformed quantum bracket, or;

(2) $[\partial]_q O=[O,H]$, where the observable evolves under symmetric q-derivative and non-deformed quantum bracket.

The author rightly notes that the most interesting case when functional time-evolution is connected with the deformed quantum bracket is also very difficult to solve. Apart from technical difficulties in solving the equation of fully deformed systems in the form $\partial_q O = \{O, H\}_q$ or $\bar{\partial}_q O = \{O, H\}_q$ the question arises of how to construct the deformed quantum bracket. We think that the answer can be found in a careful analysis of classical deformed systems and deformed Poisson brackets resulting from the functional evolution of systems depending on q-derivatives.

This section is divided into two parts; for systems with discrete and continuous time.

3.1. Deformed mechanical systems with discrete time

Let us start with the following functional of action

$$A = \int_{t_0}^{t_1} L[x(\tau), \partial_q x(\tau)] \, \mathrm{d}\mu_q(\tau) \tag{29}$$

where we have used the measure defined by (10), (11). We shall be dealing with the functions, which are in fact equivalence classes from $\Re[t_0, t_1]/\approx$ and can be identified with pairs of sequences $\{\{f(q^{2n}t_0)\}_{n=0}^{\infty}, \{f(q^{2n}t_1)\}_{n=0}^{\infty}\}$. For such functions the following lemma can be easily checked:

Lemma. If the integral

$$\int_{t_0}^{t_1} f(\tau)\eta(\tau) \, \mathrm{d}\mu_q(\tau) = 0$$

for any function $\eta \in \Re[t_0, t_1]/\approx$ vanishing for t_0 and t_1 then the function $f = 0$ for $t_j q^{2k}$, $j = 1, 2$ and $k \geq 1$.

This is an analogue of the standard lemma used in the derivation of Euler–Lagrange equations for the stationary point of a functional. Adding the variation $\alpha\eta(t)$ to variables x we get the following variation of action depending on arbitrary function η and constant α:

$$\delta A(\alpha, \eta) = \int_{t_0}^{t_1} L[x + \alpha\eta, \partial_q x + \alpha \partial_q \eta](\tau) \, \mathrm{d}\mu_q(\tau) - \int_{t_0}^{t_1} L[x, \partial_q x](\tau) \, \mathrm{d}\mu_q(\tau). \tag{30}$$

Let us analyse the derivative of variation with respect to parameter α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \delta A(\alpha, \eta) = \int_{t_0}^{t_1} \left[\frac{\delta L}{\delta x} + \partial_q^+ \frac{\delta L}{\delta \partial_q x} \right](\tau)\eta(\tau) \, \mathrm{d}\mu_q(\tau) = 0. \tag{31}$$

This equation is a necessary, and sufficient, condition for the stationary point of the functional A. According to the lemma it implies that variable x must fulfil the following deformed Euler–Lagrange equations:

$$\frac{\delta L}{\delta x} + \partial_q^+ \frac{\delta L}{\delta \partial_q x} = 0 \qquad \text{for } t_j q^{2k} \qquad j = 1, 2 \qquad k \geq 1. \tag{32}$$

It is clear that for systems with many variables depending on time we have the extended set of equations:

$$\frac{\delta L}{\delta x_i} + \partial_q^+ \frac{\delta L}{\delta \partial_q x_i} = 0 \qquad i = 1, \ldots, n \qquad t_j q^{2k} \qquad j = 1, 2 \qquad k \geq 1. \tag{33}$$

As an example of such equations we can derive them for the Lagrangian of deformed classical oscillator

$$
\begin{aligned}
&L = \partial_q x \partial_q x/2 + m^2 x^2/2 \\
&[\partial_q^+ \partial_q + m^2]x = 0.
\end{aligned}
\tag{34}
$$

The solution is the equivalence class of linear combination of the known Jackson exponential functions [12] where constants are calculated from the boundary conditions $x(t_0) = x(t_1) = 0$.

Note, also, that for such systems the Lagrangian linear in derivatives can be considered without using spinor variables

$$L = x\partial_q x + mx^2/2. \tag{35}$$

The Euler-Lagrange equation looks as follows

$$[\partial_q + \partial_q^+ + m]x = 0 \tag{36}$$

and after transformation we get the recurrence formula for its solution in the $q^{2n}t_0$

$$x(q^2 q^{2n} t_0) = x(q^{2n} t_0)[2 - m(q^2-1)q^{2n}t_0] - x(q^{-2}q^{2n}t_0) \qquad n \geqslant 1$$

and similar expression for $q^{2n}t_1$.

The next step is construction of the Hamiltonian. We define the q-deformed momentum as

$$p = \delta L/\delta\partial_q x. \tag{37}$$

Then we pass to the Hamiltonian of the system

$$H = p\partial_q x - L. \tag{38}$$

From the Hamiltonian and Euler-Lagrange equations we are able to derive evolution equations for phase-space variables

$$\partial_q x = \delta H/\delta p \qquad \partial_q p = -q^2 \zeta_{q^2} \delta H/\delta x. \tag{39}$$

These equations should determine the evolution of any function of phase-space variables. The q-derivative of such a function can be divided into two parts

$$
\begin{aligned}
&\partial_q u[x(\tau), p(\tau)] \\
&\quad = (1-a)[\partial_{x\circ\kappa_{q^2}\circ x^{-1}} u[p(q^2\tau), x(\tau)]\partial_q x(\tau) + \partial_{p\circ\kappa_{q^2}\circ p^{-1}} u[p(\tau), x(\tau)]\partial_q p(\tau)] \\
&\qquad + a[\partial_{x\circ\kappa_{q^2}\circ x^{-1}} u[p(\tau), x(\tau)]\partial_q x(\tau) + \partial_{p\circ\kappa_{q^2}\circ p^{-1}} u[p(\tau), x(q^2\tau)]\partial_q p(\tau)].
\end{aligned}
\tag{40}
$$

In this way we have obtained a formula analogous to the full derivative known in real analysis. Using formulae for derivatives (39) we get the classical evolution bracket we aimed for

$$
\begin{aligned}
&\partial_q u[x(\tau), p(\tau)] = \{u, H\}_q^a \\
&\quad = (1-a)\left[\partial_{x\circ\kappa_{q^2}\circ x} - 1u[p(q^2\tau), x(\tau)]\frac{\delta H}{\delta p}\right. \\
&\qquad \left. - q^2 \partial_{p\circ\kappa_{q^2}\circ p} - 1u[p(\tau), x(\tau)]\zeta_{q^2}\frac{\delta H}{\delta x}\right] \\
&\qquad + a\left[\partial_{x\circ\kappa_{q^2}\circ x} - 1u[p(\tau), x(\tau)]\frac{\delta H}{\delta p} - q^2\partial_{p\circ\kappa_{q^2}\circ p} - 1u[p(\tau), x(q^2\tau)]\zeta_{q^2}\frac{\delta H}{\delta x}\right].
\end{aligned}
\tag{41}
$$

For a pair of arbitrary functions C and D we have

$$\{C, D\}_q^a = (1-a)\left[\partial_{x\circ\kappa_{q^2}\circ x} - 1C[p(q^2\tau), x(\tau)]\frac{\delta D}{\delta p} - q^2\partial_{p\circ\kappa_{q^2}\circ p} - 1C[p(\tau), x(\tau)]\zeta_{q^2}\frac{\delta D}{\delta x}\right]$$
$$+ a\left[\partial_{x\circ\kappa_{q^2}\circ x} - 1C[p(\tau), x(\tau)]\frac{\delta D}{\delta p}\right.$$
$$\left. - q^2\partial_{p\circ\kappa_{q^2}\circ p} - 1C[p(\tau), x(q^2\tau)]\zeta_{q^2}\frac{\delta D}{\delta x}\right]. \tag{42}$$

In fact this is a one-parameter family of brackets but the evolution does not depend on parameter a, as, differentiating formula (41) with respect to it, we get the following identity

$$\frac{\delta H}{\delta p}(p(q^2\tau) - p(\tau)) + q^2\zeta_{q^2}\frac{\delta H}{\delta x}(x(q^2\tau) - x(\tau)) \equiv 0. \tag{43}$$

The variable x and its momentum give the following evolution brackets

$$\{p, x\}_q^a = -q^2 \qquad \{x, p\}_q^a = 1. \tag{44}$$

As we see these equalities are also independent from parameter a. It is easy to check that there exists a whole class of functions for which the evolution bracket does not depend on a. They should fulfil equations which can be derived similarly to (43). The function on the left side should fulfil

$$C[p(q^2\tau), x(q^2\tau)] - C[p(q^2\tau), x(\tau)] - C[p(\tau), x(q^2\tau)] + C[p(\tau), x(\tau)] = 0 \tag{45}$$

or alternatively for the right function we get

$$\frac{\delta D}{\delta p}(p(q^2\tau) - p(\tau)) + q^2\zeta_{q^2}\frac{\delta D}{\delta x}(x(q^2\tau) - x(\tau)) = 0. \tag{46}$$

Let us quote a few important features of constructed evolution brackets:

(i) When $q \to 1$ the evolution bracket becomes the classical Poisson bracket.

(ii) It is neither symmetric nor anti-symmetric. The commutation relations of variable x and its momentum are consistent with that of the creation and annihilation operators for $SU_q(2)$ algebra.

(iii) The Hamiltonian (38) is, in most general cases, not constant in time. However setting it as a function of x or p only we get the conserved Hamiltonian. For general cases we must solve the functional equation

$$\partial_q u = \{u, H\}_q^a = 0. \tag{47}$$

We have solved this equation for two particular cases:

(1) If the Hamiltonian is a sum of parts depending on one phase-space variable

$$H = H_1(x) + H_2(p).$$

The constant in time modification of the Hamiltonian is as follows

$$\begin{aligned} H_q &= \int_{p(0)}^{p(t)} \frac{\delta H}{\delta p}[p^{-1}(\tau)]\,\mathrm{d}\mu_{p\circ\kappa_{q^2}\circ p} - 1(\tau) + q^2\int_{x(0)}^{x(t)} \zeta_{q^2}\frac{\delta H}{\delta x}(\tau)\,\mathrm{d}\mu_{x\circ\kappa_{q^2}\circ x} - 1(\tau) \\ x(0) &= \lim_{n\to\infty}(x\circ\kappa_{q^2}\circ x^{-1})^n(\tau) \qquad p(0) = \lim_{n\to\infty}(p\circ\kappa_{q^2}\circ p^{-1})^n(\tau). \end{aligned} \tag{48}$$

(2) If the Hamiltonian is given as a product of functions depending on one variable

$$H = H_1(x)\cdot H_2(p)$$

we have the constant of evolution

$$H_q = \int_{p(0)}^{p(t)} \left[\frac{\delta H_2}{\delta p} \Big/ \zeta_q^2 H_2 \right] [p^{-1}(\tau)] \, \mathrm{d}\mu_{p \circ \kappa_{q^2} \circ p} - 1(\tau) + q^2$$
$$\times \int_{x(0)}^{x(t)} \left[\zeta_{q^2} \frac{\delta H_1}{\delta x} \Big/ H_1 \right] [p^{-1}(\tau)] \, \mathrm{d}\mu_{x \circ \kappa_{q^2} \circ x} - 1(\tau). \tag{49}$$

All the above calculations can be straightforwardly extended to systems with many variables. Similarly to our simple model the result is a many-parameter family of evolution brackets. The evolution of functions and brackets of phase-space variables does not depend on parameters. The procedure is analogous to that presented above so here we quote only the results. The q-derivative of function u, depending on many variables after using Euler–Lagrange equations (33) and extending the Hamiltonian (38), can be written in the following form:

$$\partial_q u[p(t), x(t)] = \partial_q u[f(t)] = \sum_{\substack{\text{perm}\\(k)}} a_{(k)} \sum_{i=1}^{2n} \partial_{f_{k_i} \circ \kappa_{q^2} \circ f_{k_i}^{-1}} u[f^{k_i}(t)] \cdot (-q^2 \zeta_{q^2})^{s_i} \frac{\delta H}{\delta f_{k_i}}$$

$$\begin{aligned}
&s_i = 0 \quad \text{when} \quad k_i > n \\
&s_i = 1 \quad \text{when} \quad k_i \leqslant n \\
&f^{k_1} = [f_i(q^{2(1-\delta_{k_1 i})} t)] \qquad i = 1, \ldots, 2n \\
&f^{k_2} = [f_i(q^{2(1-\delta_{k_1 i} - \delta_{k_2 i})} t)] \qquad i = 1, \ldots, 2n \\
&f^{k_r} = [f_i(q^{2(1-\Sigma_{j<r} \delta_{k_j i})} t)] \qquad i = 1, \ldots, 2n
\end{aligned} \tag{50}$$

where

$$\sum_{\substack{\text{perm}\\(k)}} a_{(k)} = 1. \tag{51}$$

The deformed evolution bracket derived similarly as in the simpler case depends now on $(2n)! - 1$ parameters

$$\{C, D\}_q^{[a]} = \sum_{\substack{\text{perm}\\(k)}} a_{(k)} \sum_{i=1}^{2n} \partial_{f_{k_i} \circ \kappa_{q^2} \circ f_{k_i}^{-1}} C[f^{k_i}(t)] \cdot (-q^2 \zeta_{q^2})^{s_i} \delta D / \delta f_{k_i}. \tag{52}$$

It is simple to check that the evolution brackets of the phase-space variables do not depend on parameters

$$\{p_i, x_j\}_q^{[a]} = -q^2 \delta_{ij} \qquad \{x_i, p_j\}_q^{[a]} = \delta_{ij} \tag{53}$$

where we have only written the non-vanishing evolution brackets. Let us notice that the remarks about the properties of the evolution bracket also hold in this general case.

In the presented analysis, only the systems with q-derivatives of variables were considered. As we have formulae for conjugation of arbitrary functional derivatives we are ready to extend the calculations. However, the independent of time evolution bracket would be obtained for ϕ-linear function only. For other functions we have brackets between the variable x and its momentum depending on time. We hope to present these results in a subsequent paper.

3.2. *Deformed systems with continuous time*

According to the formulae of conjugation for functional derivatives, as long as we restrict ourselves to function ϕ-linear, we can build the action over the continuous, infinite interval of time

$$A=\int_0^\infty L[x_i(\tau), \partial_q x_i(\tau)]\,\mathrm{d}\tau. \tag{54}$$

In this model we assume x_i to be continuous functions on $[0,\infty]$. The expressions derived in section 1.2 for conjugation with respect to Lebesgue measure are the last needed step for construction of Euler–Lagrange equations of a stationary point. They have identical form to the previously given equation (33) for systems with discrete time but are now fulfilled for $\tau\in[0,\infty]$

$$\begin{aligned} &\frac{\delta L}{\delta x_i}+\partial_q^+\frac{\delta L}{\delta\partial_q x_i}=0 \qquad i=1,\ldots,n\\ &H=p_i\partial_q x_i-L. \end{aligned} \tag{55}$$

Here we summarize the results as the procedure is similar to that used in the preceding section. We have the following evolution equation, identical in form with the equation for systems with discrete time

$$\partial_q u[p(t),x(t)]=\partial_q u[f(t)]=\{u,H\}_q^{[a]}=\sum_{\substack{\text{perm}\\(k)}} a_{(k)}\sum_{i=1}^{2n}\partial_{f_{k_i}\circ\kappa_{q^2}\circ f_{k_i}^{-1}}u[f^{k_i}(t)]\cdot(-q^2\zeta_{q^2})^{s_i}\frac{\delta H}{\delta f_{k_i}}$$

$$\begin{aligned} &s_i=0 \quad \text{when} \quad k_i>n\\ &s_i=1 \quad \text{when} \quad k_i\leqslant n\\ &f^{k_1}=[f_i(q^{2(1-\delta_{k_1 i})}t)] \qquad i=1,\ldots,2n\\ &f^{k_2}=[f_i(q^{2(1-\delta_{k_1 i}-\delta_{k_2 i})}t)] \qquad i=1,\ldots,2n\\ &f^{k_r}=[f_i(q^{2(1-\Sigma_{j<r}\delta_{k_j i})}t)] \qquad i=1,\ldots,2n \end{aligned} \tag{56}$$

where

$$\sum_{\substack{\text{perm}\\(k)}} a_{(k)}=1.$$

Formulae for evolution brackets of phase-space variables coincide with (53)

$$\{C,D\}_q^{[a]}=\sum_{\substack{\text{perm}\\(k)}} a_{(k)}\sum_{i=1}^{2n}\partial_{f_{k_i}\circ\kappa_{q^2}\circ f_{k_i}^{-1}}C[f^{k_i}(t)]\cdot(-q^2\zeta_{q^2})^{s_i}\frac{\delta D}{\delta f_{k_i}}. \tag{57}$$

Also the remarks about constants of evolution and the solutions (48), (49) for particular cases of Hamiltonians apply to the model with continuous time.

We end this section with an example of a system with constraints which can be considered both as continuous or discrete time models.

3.3. *Example*

Let us start with the following linear Lagrangian derivative of complex field

$$L=x\partial_q^+\bar{x}+x\partial_q\bar{x}+V(x,\bar{x}). \tag{58}$$

Deriving the momenta we get the constraints

$$p=\frac{\delta L}{\delta\partial_q x}=\bar{x} \qquad \bar{p}=\frac{\delta L}{\delta\partial_q^+\bar{x}}=x$$

$$\chi_1=p-\bar{x} \qquad \chi_2=\bar{p}-x$$

which are second class and conserved in time.

So we should modify evolution brackets (52), (57) in a similar manner to the procedure of Dirac quantization of systems with second-class constraints [17, 18].

Modified evolution brackets look as follows

$$\{C,D\}_q^D=\{C,D\}_q^a-\{C,\chi_\alpha\}_q^a\{\chi_\alpha,\chi_\beta\}_q^{a-1}{}_{\alpha\beta}\{\chi_\beta,D\}_q^a \tag{59}$$

and for variables and their momenta we have

$$\{p,p\}_q^D=\{\bar{x},p\}_q^D=\{p,\bar{x}\}_q^D=\{\bar{x},\bar{x}\}_q^D=\{\bar{p},\bar{p}\}_q^D=\{x,\bar{p}\}_q^D=\{\bar{p},x\}_q^D=\{x,x\}_q^D=0$$

$$\{p,\bar{p}\}_q^D=\{\bar{x},\bar{p}\}_q^D=\{p,x\}_q^D=\{\bar{x},x\}_q^D=-q^2/2$$

$$\{\bar{p},p\}_q^D=\{x,p\}_q^D=\{\bar{p},\bar{x}\}_q^D=\{x,\bar{x}\}_q^D=1/2.$$

It is interesting to note that in our example physical variables (on the submanifold given by constraints) can be chosen to fulfil symmetric or anti-symmetric modified evolution brackets

$$\begin{aligned}\{p+\alpha x,\bar{p}+\beta\bar{x}\}_D&=-q^2/2+\alpha\beta/2\\ \{\bar{p}+\beta\bar{x},p+\alpha x\}_D&=1/2+\alpha\beta(-q^2/2)\end{aligned} \tag{60}$$

accordingly for $\alpha\beta=1$ or $\alpha\beta=-1$.

4. Final remarks

In this paper we have studied the properties of general functional derivatives and presented conjugation formulae for discrete and continuous time functionals. We applied the evaluated methods to classical mechanical systems depending on q-derivatives. Assuming that q-evolution of functions on phase-space is determined by an evolution bracket with a Hamiltonian, we derived the explicit form of many-parameters family of evolution brackets. We showed that the evolution as well as canonical evolution brackets of phase-space variables do not depend on parameters. Investigated models have non-symmetric evolution brackets resembling deformed commutators of independent oscillator realizations of $SU_q(n)$. The characteristic feature of such systems are non-conserved Hamiltonians. However we were able to construct, for certain classes of Hamiltonians, constants of evolution. It is also clear from our construction how we would evolve systems dependent on general ϕ-derivatives of variables. For linear functions we get formulae identical to that presented in section 2 while the arbitrary function ϕ implies time-dependence of evolution brackets on phase-space.

Many interesting problems for such models remain unsolved. One of them is the extension, to complex functions, of complex time. When q is complex the calculations can probably be easily generalized both in discrete and continuous time realizations, if it is not a root of unity. For q-root-of-unity the inverse operator cannot be uniquely determined. This fact would in our opinion provide a serious difficulty in the construction of the discrete time model.

Another open question is quantization of such models. So far we do not know any reliable prescription of how to pass from classical to quantum evolution bracket.

Finally the problem arises whether models depending on different functional derivatives (for example different q_i-derivatives) can be treated similarly to the presented systems depending only on the one functional derivative.

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